I. The Regression Approach to Portfolio Analysis

In this paper a riskless asset is assumed available for both borrowing and lending in each period. *Excess returns* are calculated by subtracting the return of this riskless asset from the total return.[6] There are K risky assets indexed by $k = 1,\ldots,K$. The excess returns on the K assets in some period t in $(1,\ldots,T)$ are denoted by the K elements of the vector $\mathbf{x}_t$:

$$\mathbf{x}_t' = [x_{1t},\ldots x_{kt},\ldots,x_{Kt}]. \tag{1}$$

The T observations of excess returns are contained in the $T \times K$ matrix $\mathbf{X}$:

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1' \\ \vdots \\ \mathbf{x}_T' \end{bmatrix}. \tag{2}$$

Note that a portfolio of risky assets and a riskless asset has an excess return that is determined solely by the weights and excess returns of the risky assets. Thus, given a K-vector of risky asset weights $\mathbf{b}$, the excess return of this portfolio in period t is simply $\mathbf{x}_t'\mathbf{b}$, where the weights in $\mathbf{b}$ need not sum to one.

Let $\mathbf{l}$ represent a vector of ones with length conforming to the rules of matrix algebra. Viewed as a portfolio excess return, the T-vector of ones $\mathbf{l}$ is highly desirable as it has positive excess return with zero sample standard deviation. The regression approach to portfolio selection[7] is based on minimizing the squared deviations between the excess returns on a constructed portfolio and the excess returns in $\mathbf{l}$. This minimization problem can be performed using an artificial ordinary least squares (OLS) regression and the following proposition states that such a regression recovers the weights of a sample efficient portfolio.

THEOREM 1: *OLS regression of a constant* **1** *onto a set of asset's excess returns* **X**, *without an intercept term,*

$$\underset{(T\times 1)}{\mathbf{1}} = \underset{(T\times k)\,(k\times 1)}{\mathbf{X}\mathbf{b}} + \underset{(T\times 1)}{\mathbf{u}}, \tag{3}$$

results in an estimated coefficient vector

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{1}, \tag{4}$$

that is a set of risky-asset-only portfolio weights for a sample efficient portfolio. The scaled (so that weights sum to one) coefficient vector $\hat{\mathbf{b}}/\mathbf{1}'\hat{\mathbf{b}}$ *is thus the familiar tangency portfolio*

$$\frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{\mathbf{1}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}, \tag{5}$$

derived from quadratic programming, where the sample mean $\bar{\mathbf{x}} = \mathbf{X}'\mathbf{1}/T$, *and the (maximum likelihood) sample covariance* $\bar{\mathbf{\Sigma}} = (\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}')'(\mathbf{X} - \mathbf{1}\bar{\mathbf{x}}')/T$, *are used as parameters.*

Proof: Using the *updating formula* for an inverse matrix,[8] express the coefficient vector $\hat{\mathbf{b}}$ from the regression in equation (3) in terms of the sample mean $\bar{\mathbf{x}}$ and sample covariance $\bar{\mathbf{\Sigma}}$:

$$\begin{aligned}\hat{\mathbf{b}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{1}\\ &= (\bar{\mathbf{\Sigma}} + \bar{\mathbf{x}}\bar{\mathbf{x}}')^{-1}\bar{\mathbf{x}}\\ &= \left(\bar{\mathbf{\Sigma}}^{-1} - \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}\bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}}{1 + \bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}\right)\bar{\mathbf{x}}\\ &= \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{1 + \bar{\mathbf{x}}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}.\end{aligned} \tag{6}$$

Scaling $\hat{\mathbf{b}}$ so that the coefficients sum to one results in the tangency portfolio

$$\frac{\hat{\mathbf{b}}}{\mathbf{1}'\hat{\mathbf{b}}} = \frac{\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}}{\mathbf{1}'\bar{\mathbf{\Sigma}}^{-1}\bar{\mathbf{x}}} \tag{7}$$

when sample means and covariances are used as parameters. Q.E.D.

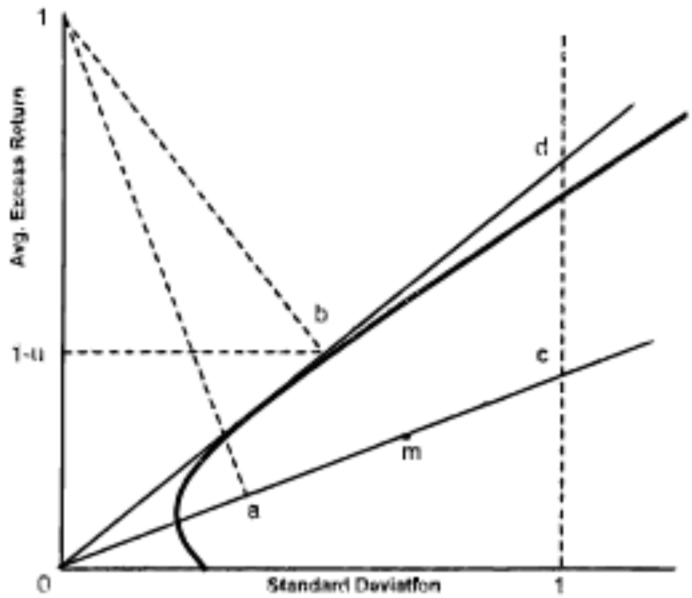

Figure 1. Sample mean standard deviation diagram. The point b is the point on the line 0d that is closest to the point (0,1) and the point a is the point on the line 0m that is closest to (0,1).

The regression in equation (3) is unusual. There is no intercept, the dependent variable is nonstochastic, and the residual vector **u** is correlated with the regressors, which are stochastic. However, the regression has a simple interpretation: The dependent variable **1** is a sample counterpart to arbitrage profits—positive excess return with zero standard deviation; the coefficients **b** represent the weights on risky assets in the portfolio; **X***b* represents excess returns on this portfolio; and the residual vector **u** shows deviations in this portfolio's return from **1**.

The estimated portfolio weights $\hat{\mathbf{b}}$ produce a portfolio return vector that is closest in terms of least squares distance to the arbitrage return vector **1**. This least squares distance can be illustrated using the familiar mean-standard deviation diagram. The feasible set, constructed from the sample mean and (maximum-likelihood) sample covariance, has an efficient boundary shown by the line 0d from the origin passing through the tangency portfolio (Figure 1). The arbitrage return vector **1** is located at the point (0,1).